On the Strong Duality in Continuous-time and Discrete-time Linear Quadratic Regulators

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***Abstract*— This paper revisits the strong duality in the linear quadratic regulator (LQR) for continuous-time and discrete-time systems, and explores its interconnection with typical assumptions and the uniqueness of primal-dual solutions. Using a linear operator Ψ, we formulate a common nonconvex LQR problem that captures both time domains. We then derive its Lagrange dual problem and establish the strong duality via a rank-constrained tight semidefinite program (SDP) relaxation. Further, we show that the primal-dual optimal solutions to the SDP relaxation, after dropping the rank constraint, recover the classical algebraic Riccati equations and optimal feedback gains in a constructive manner. The dual derivation and strong duality analysis exploit the properties of the linear operator and its adjoint, revealing a structural symmetry between the two time domains.**

***Index Terms*— LQR, strong duality, semidefinite program (SDP), algebraic Riccati equation.**

I. INTRODUCTION

The Linear Quadratic Regulator (LQR) is arguably the most fundamental optimal control problem [1], with significant theoretical importance and practical applications. The goal is to design an optimal controller for a linear dynamical system by minimizing a quadratic cost accumulated along the system trajectory. The LQR problem is known to enjoy many favorable theoretical features. For example, an elegant fact is that in the infinite-horizon formulation, the optimal control policy is linear and static of the form $u = Kx$. It also admits a convex semidefinite program (SDP) reformulation through a change of variables. Moreover, a nonconvex LQR formulation in the policy space exhibits a strongly convex-like landscape, known as *gradient dominance* [2]–[5]. Recently, as an alternative perspective, Bamieh [6] and our recent work [3] revisited LQR using an SDP and duality strategy [7], formalizing a duality-based derivation of the optimal gain. Our work [3] also shows that for continuous-time systems, a nonconvex LQR can be viewed as a quadratically constrained quadratic programming (QCQP) with a tight SDP relaxation and zero duality gap.

It is known that both continuous-time and discrete-time LQR problems satisfy the above favorable properties almost in the same way. However, the majority of existing studies have dealt with each time domain separately (including the duality analyses [3], [6], [7]). While such a treatment is natural and simplifies the exposition, this makes the symmetry and subtlety between the two time domains less explicit. Toward closing this gap, this paper revisits the strong duality and develops a common framework for continuous-time and discrete-time LQRs. The topic of strong duality in LQR is not new, and there exist many duality-based analyses and synthesis for linear control [8]–[14]. To our knowledge, continuous-time and discrete-time LQRs have been analyzed separately via different proof techniques [3], [6], [8], [15], [16]. In these works, slightly different assumptions were made, particularly in terms of the stochasticity in initial conditions and the uniqueness of the optimal gain. The work [16] showed strong duality for a discrete-time LQR with a risk constraint, but it requires stronger assumptions on the weight and covariance matrices. In contrast, we impose only basic and standard assumptions, which cover LQRs with both stochastic and deterministic initial conditions. Further, we work with a common LQR formulation for two time domains using a linear operator Ψ [3], inspired by earlier works for linear matrix inequalities (LMIs) [10], [13], [17]. While this formulation is nonconvex, we can derive the Lagrange dual and an SDP relaxation, similarly to Shor's relaxation of QCQPs [18]. We then demonstrate the strong duality by showing the tightness of this relaxation and by utilizing the SDP duality theory.

Our results provide the following insights. First, our results clarify that in the strong duality analysis, algebraic Riccati equations (AREs), and optimal LQR gains, the distinction between the two time domains for LQR is purely encoded in the linear operator Ψ. In our proof, similar to Shor's SDP relaxation for QCQPs, we derive a common SDP relaxation from a rank-constrained form of the LQR. The SDP relaxation turns out to be lossless, and the desirable strong duality follows from the analysis of its Karush-Kuhn-Tucker (KKT) conditions. Second, we clarify that strong duality holds for both time domains under highly mild assumptions, covering both deterministic and stochastic initial conditions. Compared with [8], [15], [16], our result only requires the initial covariance matrix $\mathbb{E}[x_0x_0^\mathsf{T}]$ to be positive semidefinite. This condition includes the deterministic LQR as a special case, is sufficient for strong duality, and separates strong duality from the uniqueness of the optimal gain. This also indicates that strong duality is a less fragile manifestation of benign nonconvexity than the gradient dominance [2]–[5]. Finally, our rank-structure-based proof identifies the optimal gains *constructively*, without assuming their form in advance. This may be viewed as a benefit of our approach compared

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with [6], [15], and further enables us to clarify when the optimal LQR gain is unique.

The rest of this paper is organized as follows. Section II formulates the LQR problems and presents the problem statement. Then, we state our main result in Section III, and present its proof in Section IV. Finally, Section V concludes the paper.

II. PROBLEM FORMULATION

Here, we formulate the LQR problems for continuous-time and discrete-time systems, and provide our problem statement.

A. Continuous-time and discrete-time LQR problems

1) Continuous-time LQR: Consider a continuous-time linear time-invariant (LTI) system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the input. Let $x(0) = x_0 \in \mathbb{R}^n$, and assume that x_0 is a random variable with zero mean and covariance $W = \mathbb{E}[x_0 x_0^{\mathsf{T}}] \in \mathbb{S}_+^n$, where $\mathbb{S}_+^n$ is the set of $n \times n$ positive semidefinite matrices.

Given weight matrices $Q \in \mathbb{S}_+^n$ and $R \in \mathbb{S}_{++}^m$ (denoting the set of $m \times m$ positive definite matrices), consider the infinite-horizon LQR problem to minimize the cost $J_{\mathtt{ct}}(K)$ below:

$$\min_K \; J_{\mathtt{ct}}(K) := \mathbb{E}\left[\int_0^\infty \left(x(t)^{\mathsf{T}} Q x(t) + u(t)^{\mathsf{T}} R u(t)\right) dt\right] \tag{2}$$

over all static state-feedback controllers of the form $u(t) = Kx(t)$, where $K \in \mathbb{R}^{m\times n}$ belongs to the set of stabilizing gains $\mathcal{K}_{\mathtt{ct}} := \{K \mid \mathrm{Re}(\lambda_i(A+BK)) < 0, \forall i = 1,\ldots,n\}$. It is well known that this class contains the continuous-time LQR optimal controller. Hence, (2) is a finite-dimensional problem over $K \in \mathcal{K}_{\mathtt{ct}} \subset \mathbb{R}^{m\times n}$. Note, however, that both the objective $J_{\mathtt{ct}}$ and the feasible set $\mathcal{K}_{\mathtt{ct}}$ are nonconvex.

2) Discrete-time LQR: Consider a discrete-time LTI system

$$x_{t+1} = Ax_t + Bu_t, \tag{3}$$

where $x_t \in \mathbb{R}^n$ is the state and $u_t \in \mathbb{R}^m$ is the input. As in the continuous-time case, let $x_0 \in \mathbb{R}^n$ be a random initial state with zero mean and covariance $W = \mathbb{E}[x_0 x_0^{\mathsf{T}}] \in \mathbb{S}_+^n$.

Given weight matrices $Q \in \mathbb{S}_+^n$ and $R \in \mathbb{S}_{++}^m$, the infinite-horizon discrete-time LQR problem is

$$\min_K \quad J_{\mathtt{dt}}(K) := \mathbb{E}\left[\sum_{t=0}^\infty \left(x_t^{\mathsf{T}} Q x_t + u_t^{\mathsf{T}} R u_t\right)\right] \tag{4}$$

over all static state-feedback controllers $u_t = Kx_t$, where K belongs to the set of Schur stabilizing gains $\mathcal{K}_{\mathtt{dt}} := \{K \in \mathbb{R}^{m\times n} \mid \rho(A+BK) < 1\}$ with $\rho(\cdot)$ denoting the spectral radius. This class is also rich enough to contain the discrete-time LQR optimal gain. Thus, (4) is again a finite-dimensional problem over $K \in \mathcal{K}_{\mathtt{dt}} \subset \mathbb{R}^{m\times n}$. As in continuous time, this problem is nonconvex, and so is the set $\mathcal{K}_{\mathtt{dt}}$.

Throughout the paper, we make a standard assumption that ensures the existence of the optimal LQR gain $K^\star = K^\star_{\mathtt{ct}}$ (in continuous time) or $K^\star_{\mathtt{dt}}$ (in discrete time).

Assumption 1: (A, B) is stabilizable and $(Q^{1/2}, A)$ is detectable.[1] The covariance matrix W satisfies $W \in \mathbb{S}_+^n \setminus \{0\}$.

[1] Stabilizability and detectability in the two time domains indicate essentially the same concepts, but their mathematical characterizations slightly differ from each other; see [19].

Note that Assumption 1 covers the standard LQR with a deterministic initial condition x_0, which corresponds to $W = x_0 x_0^{\mathsf{T}}$. This assumption is also connected to the *stochastic LQR* with a Gaussian process noise and $\mathcal{H}_2$ control [19], [20].

B. A unified reformulation of LQR

It is known that, after a suitable reformulation using Lyapunov equations, both continuous-time and discrete-time LQRs admit an operator-based common form. We begin with the standard Lyapunov representation of the LQR cost. For continuous time, if K belongs to $\mathcal{K}_{\mathtt{ct}}$ (i.e., stabilizing), then

$$J_{\mathtt{ct}}(K) = \langle Q + K^{\mathsf{T}} R K, X\rangle,$$

where $\langle S, T\rangle := \mathbf{Tr}(S^{\mathsf{T}} T)$ denotes the inner product of two matrices, and X uniquely solves the Lyapunov equation:

$$A_K X + X A_K^{\mathsf{T}} + W = 0, \tag{5}$$

with $A_K := A + BK$ being the closed-loop matrix. The matrix X can be interpreted as the cumulative covariance $X = \int_0^\infty x(t)x(t)^{\mathsf{T}} dt$. Similarly, for discrete time, if K belongs to $\mathcal{K}_{\mathtt{dt}}$ (i.e., stabilizing), then

$$J_{\mathtt{dt}}(K) = \langle Q + K^{\mathsf{T}} R K, X\rangle,$$

where X uniquely solves the discrete-time Lyapunov equation:

$$A_K X A_K^{\mathsf{T}} - X + W = 0. \tag{6}$$

In this case, the matrix X corresponds to $X = \sum_{t=0}^\infty x_t x_t^{\mathsf{T}}$. These representations are standard; see, e.g., [2]–[4], [20].

The two expressions already share the same objective $\langle Q + K^{\mathsf{T}} R K, X\rangle$. Their difference lies only in the Lyapunov equations (5) and (6): in continuous time, the system's evolution is captured by the derivative

$$\int_0^\infty \frac{d}{dt}(x(t)x(t)^{\mathsf{T}})dt = \int_0^\infty (\dot{x}(t)x(t)^{\mathsf{T}} + x(t)\dot{x}(t)^{\mathsf{T}})dt,$$

leading to $A_K X + X A_K^{\mathsf{T}}$, whereas in discrete time we use a one-step difference $\sum_{t=0}^\infty (x_{t+1}x_{t+1} - x_t x_t)$, leading to $A_K X A_K^{\mathsf{T}} - X$. This purely algebraic distinction motivates the following common form:

$$\Psi\left(\begin{bmatrix} A_K \\ I \end{bmatrix} X \begin{bmatrix} A_K \\ I \end{bmatrix}^{\mathsf{T}}\right) + W = 0, \tag{7}$$

where $\Psi : \mathbb{S}^{2n} \to \mathbb{S}^n$ is the linear operator defined by $\Psi = \Psi_{\mathtt{ct}}$ in continuous time and $\Psi = \Psi_{\mathtt{dt}}$ in discrete time, with

$$\Psi_{\mathtt{ct}}\left(\begin{bmatrix} F & G \\ G^{\mathsf{T}} & H \end{bmatrix}\right) = G + G^{\mathsf{T}}, \Psi_{\mathtt{dt}}\left(\begin{bmatrix} F & G \\ G^{\mathsf{T}} & H \end{bmatrix}\right) = F - H.$$

Then (5) is equivalent to (7) with $\Psi = \Psi_{\mathtt{ct}}$, while (6) is equivalent to (7) with $\Psi = \Psi_{\mathtt{dt}}$. This operator-based unification was used in our recent work [4], and related ideas also appear in earlier LMI formulations such as [10], [13], [17].

With (7), we introduce the following reformulation:

$$\begin{aligned} p^\star = \min_{K,X} \quad & \langle Q + K^{\mathsf{T}} R K, X\rangle \\ \text{subject to} \quad & (7), \quad X \succeq 0, \end{aligned} \tag{Primal}$$

which serves as the primal LQR problem in this paper. For each stabilizing gain K, the Lyapunov equations (5) and (6) admit a unique solution $X \succeq 0$. In (Primal), however, we treat X as an explicit decision variable and remove the stabilizing constraint, which is convenient for deriving the dual problem.

Remark 1: When $W \not\succ 0$, the feasible set of (Primal) may be larger than the set of stabilizing gains, since some non-stabilizing gains may also admit a feasible matrix $X \succeq 0$ satisfying (7). Thus, (Primal) is generally not an exact reformulation at the level of feasible points. Nevertheless, it is known that (Primal) still attains its optimum at the stabilizing gain $K^\star$. Consequently, it yields the same optimal value and recovers the same optimal controller as (2) and (4). For this reason, (Primal) can be viewed as a valid reformulation. □

C. Problem statement

As expected, the reformulated problem (Primal) remains nonconvex, since the constraint (7) still couples K and X nonlinearly. However, this formulation (Primal) satisfies strong duality under mild conditions. In continuous time, the corresponding strong duality and its connection with Riccati equations are closely related to existing results [3], [6], [9]. The discrete-time case has been studied in [15] under the assumption of $W \succ 0$.

The main purpose of this paper is to present a transparent and common proof strategy for both continuous-time and discrete-time LQRs through the operator Ψ. In particular, our goal is to show that the two time domains share the same primal-dual mechanism, the associated algebraic Riccati equations (AREs), and the optimal gains in a common way. This motivates the following problem.

Problem 1: Under Assumption 1, derive the Lagrange dual problem of (Primal), establish strong duality between the primal and dual problems, and characterize their solutions in a unified manner for continuous-time and discrete-time systems.

III. Strong duality in LQR

We here present the strong duality for the LQR in (Primal), and postpone the main proof techniques to Section IV.

A. Lagrange dual formulation

We begin with deriving the Lagrange dual of (Primal). This follows the standard process, but we need to use the adjoint operator of Ψ and its property to derive an explicit dual.

Define the Lagrangian with the Lagrange multiplier $P \in \mathbb{S}^n$:

$$\begin{aligned} L(K,X,P) =& \langle Q + K^\mathsf{T} R K, X\rangle \\ &+ \left\langle P, \Psi\left(\begin{bmatrix} A_K \\ I\end{bmatrix} X \begin{bmatrix} A_K \\ I\end{bmatrix}^\mathsf{T}\right) + W\right\rangle. \end{aligned}$$

Then, the dual problem is given as an abstract form below

$$\sup_{P\in\mathbb{S}^n} \quad d(P) \quad \text{with} \quad d(P) = \inf_{K, X\succeq 0} L(K,X,P). \tag{8}$$

We derive an explicit form for function d. First, we have

$$\begin{aligned} \langle Q + K^\mathsf{T} R K, X\rangle &= \left\langle \begin{bmatrix} Q & 0 \\ 0 & R\end{bmatrix}, \begin{bmatrix} I \\ K\end{bmatrix} X \begin{bmatrix} I \\ K\end{bmatrix}^\mathsf{T}\right\rangle, \\ \begin{bmatrix} A_K \\ I\end{bmatrix} X \begin{bmatrix} A_K \\ I\end{bmatrix}^\mathsf{T} &= \begin{bmatrix} A & B \\ I & 0\end{bmatrix}\begin{bmatrix} I \\ K\end{bmatrix} X \begin{bmatrix} I \\ K\end{bmatrix}^\mathsf{T}\begin{bmatrix} A & B \\ I & 0\end{bmatrix}^\mathsf{T}. \end{aligned} \tag{9}$$

Now, we introduce the adjoint operator Ψ^* of Ψ (i.e., $\langle \Psi(Z), P\rangle = \langle Z, \Psi^*(P)\rangle$):

$$\Psi^*_{\mathrm{ct}}(P) = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix} \otimes P, \quad \Psi^*_{\mathrm{dt}}(P) = \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix} \otimes P. \tag{10}$$

The derivation of the adjoint Ψ^* is standard, and we provide some details in Appendix A. With (9) and (10), we can rewrite the Lagrangian as

$$\begin{aligned} L(K,X,P) &\overset{(a)}{=} \left\langle \begin{bmatrix} Q & 0 \\ 0 & R\end{bmatrix}, \begin{bmatrix} I \\ K\end{bmatrix} X \begin{bmatrix} I \\ K\end{bmatrix}^\mathsf{T}\right\rangle \\ &+ \left\langle \begin{bmatrix} A & B \\ I & 0\end{bmatrix}^\mathsf{T} \Psi^*(P) \begin{bmatrix} A & B \\ I & 0\end{bmatrix}, \begin{bmatrix} I \\ K\end{bmatrix} X \begin{bmatrix} I \\ K\end{bmatrix}^\mathsf{T}\right\rangle + \langle P, W\rangle \end{aligned}$$

where (a) uses the definition of the adjoint operator Ψ^* and inner product $\langle\cdot,\cdot\rangle$. Thus, a rearrangement yields

$$L(K,X,P) = \langle P, W\rangle + \left\langle U(P), \begin{bmatrix} I \\ K\end{bmatrix} X \begin{bmatrix} I \\ K\end{bmatrix}^\mathsf{T}\right\rangle, \tag{11}$$

where we define

$$U(P) := \begin{bmatrix} A & B \\ I & 0\end{bmatrix}^\mathsf{T} \Psi^*(P) \begin{bmatrix} A & B \\ I & 0\end{bmatrix} + \begin{bmatrix} Q & 0 \\ 0 & R\end{bmatrix}. \tag{12}$$

This matrix $U(P)$ plays an important role in our subsequent discussions. Using (10), we can write a more explicit form as

$$U(P) = \begin{cases} \begin{bmatrix} A^\mathsf{T} P + PA + Q & PB \\ B^\mathsf{T} P & R\end{bmatrix} & \text{(CT)} \\ \begin{bmatrix} A^\mathsf{T} PA - P + Q & A^\mathsf{T} PB \\ B^\mathsf{T} PA & R + B^\mathsf{T} PB\end{bmatrix}, & \text{(DT)}\end{cases} \tag{13}$$

where CT and DT stand for continuous time and discrete time, respectively. From (11), we have the following result.

Lemma 1: With Assumption 1, we have

$$d(P) = \begin{cases} \langle P, W\rangle, & \text{if } U(P) \succeq 0 \\ -\infty, & \text{otherwise.}\end{cases}$$

Proof: If $U(P) \succeq 0$, it is clear from (11) that $d(P) = \min_{K,X\succeq 0} L(K,X,P) = \langle P, W\rangle$, where the minimum is achieved by $X = 0$. Otherwise, we can show $d(P) = -\infty$ by a careful eigenvalue argument. We present some details in Appendix B. ■

Consequently, the dual problem of (Primal) is expressed as

$$d^\star = \sup_{P\in\mathbb{S}^n} \quad \langle P, W\rangle \quad \text{subject to} \quad U(P) \succeq 0. \tag{Dual}$$

Note that this dual problem (Dual) is an SDP. We always have $p^\star \geq d^\star$ following the standard weak duality.

Remark 2: The matrix $U(P)$ plays an important role for LQR in both theoretical and practical ways. The inequality $U(P) \succeq 0$ is called an *algebraic Riccati inequality* (ARI) and has many classical results [21]. On the other hand, this matrix is also closely related to the model-free *Q-learning* method [22] for discrete-time systems. In particular, the Q-function for the discrete-time LQR is given by $Q(x,u) = z^\mathsf{T} U(P) z$ with $z = [x^\mathsf{T}, u^\mathsf{T}]^\mathsf{T}$ and P from the ARE.

B. Strong duality and primal and dual solutions

Despite the nonconvexity of the LQR problems, we can guarantee strong duality between (Primal) and (Dual).

Theorem 1 (Strong duality): With Assumption 1, the duality gap between (Primal) and (Dual) is zero, i.e., $p^\star = d^\star$.

The strong duality only requires the basic stabilizability and detectability assumptions and positive semidefinite $W = \mathbb{E}[x_0 x_0^\mathsf{T}]$ for both time domains, while gradient dominance [2],

[5] requires a much stronger assumption on W and has a different statement depending on the time domain [4]. Our recent work [3] focused on the continuous-time LQR, and earlier discussions can be found in [6], [8], [9], [15], [16]. In contrast, Theorem 1 applies to both continuous-time and discrete-time LQRs, since (Primal) and (Dual) encompass both time domains via the operator Ψ. Moreover, our proof provides a unified treatment by exploiting the property of Ψ.

We present a proof sketch for Theorem 1; full details are in Section IV. We first introduce a rank-constrained SDP by introducing a new matrix variable, and dropping the rank constraint yields an SDP relaxation of (Primal). This procedure is similar to Shor's SDP relaxation for QCQPs [18]. We then show that the SDP relaxation not only shares the same dual in (Dual) but also is tight, i.e., it preserves the same optimal value as (Primal). This is ensured by checking strong duality and solving the KKT equation for the SDPs. One notable feature is that the KKT solution for the primal and dual SDPs must satisfy the original rank constraint, which ensures Theorem 1.

In addition to the strong duality, we can further characterize the optimal solutions to primal and dual LQRs.

Proposition 1: Consider the primal-dual problems (Primal) and (Dual). Suppose Assumption 1 holds. We partition the matrix $U(P)$ in (12) as

$$U(P) = \begin{bmatrix} U_{11}(P) & U_{12}(P) \\ U_{12}^{\mathsf{T}}(P) & U_{22}(P) \end{bmatrix}. \tag{14}$$

Then, the following statements hold.

1) An optimal solution to (Dual) is given by the unique stabilizing solution $P^\star \succeq 0$ to the ARE:
$$U_{11}(P) - U_{12}(P)U_{22}^{-1}(P)U_{12}^{\mathsf{T}}(P) = 0. \tag{15}$$
2) An optimal solution $(K^\star, X^\star)$ to (Primal) is given by
$$K^\star = -U_{22}^{-1}(P^\star)U_{12}^{\mathsf{T}}(P^\star), \tag{16}$$
and $X^\star$ that solves the Lyapunov equation (7) for $K^\star$.

Moreover, when $W \succ 0$, the solutions above are the only ones.

This proposition reinterprets classical results on continuous-time and discrete-time LQRs [19]. After substituting (13) into (14), the ARE (15) is indeed in the familiar form

$$\begin{cases} A^{\mathsf{T}}P + PA + Q - PBR^{-1}B^{\mathsf{T}}P = 0 & \text{(CT)} \\ A^{\mathsf{T}}PA - P + Q \\ \quad -A^{\mathsf{T}}PB(R + B^{\mathsf{T}}PB)^{-1}B^{\mathsf{T}}PA = 0, & \text{(DT)} \end{cases} \tag{17}$$

and the optimal LQR gain (16) becomes

$$K^\star = \begin{cases} -R^{-1}B^{\mathsf{T}}P^\star \in \mathcal{K}_{\mathrm{ct}} & \text{(CT)} \\ -(R + B^{\mathsf{T}}P^\star B)^{-1}B^{\mathsf{T}}P^\star A \in \mathcal{K}_{\mathrm{dt}}. & \text{(DT)} \end{cases} \tag{18}$$

In other words, the usual Riccati equation (17) arises from the Schur complement of the block matrix $U(P)$. We can obtain a pair of primal optimal solutions $(K^\star, X^\star)$ from the dual optimal solution $P^\star$. One relatively less emphasized point is that the solution to the primal LQR problem (Primal) may not be unique, unless $W \succ 0$. Explicit continuous-time and discrete-time LQR instances can be found in [4, Examples 3 and 4]. We provide a proof of Proposition 1 in Section IV.

IV. Proof of Strong Duality via Rank-Constrained SDP Relaxations

Here, we establish Theorem 1 and Proposition 1 by introducing a rank-constrained SDP for (Primal) together with careful primal and dual analysis.

A. Rank-constrained SDP and its Shor's relaxation

We here derive a rank-constrained SDP form for (Primal). Dropping the rank constraint directly leads to a standard SDP, which turns out to be lossless. This process is analogous to the well-known Shor's relaxation procedure for QCQPs. We illustrate this with a simple example.

Example 1: Consider a homogeneous QCQP:

$$\begin{aligned} \min_{s \in \mathbb{R}^n} \quad & s^{\mathsf{T}}A_0 s \\ \text{subject to} \quad & s^{\mathsf{T}}A_i s + c_i \leq 0,\ i = 1, \ldots, p, \end{aligned}$$

where $A_0, A_1 \ldots, A_p \in \mathbb{S}^n$. This QCQP problem is generally nonconvex. Now, let us introduce a new matrix variable $S = ss^{\mathsf{T}} \in \mathbb{S}^n$, which is equivalent to $\operatorname{rank}(S) = 1$ and $S \succeq 0$. Thus, the QCQP problem can be equivalently rewritten as

$$\begin{aligned} \min_{S \in \mathbb{S}^n} \quad & \langle A_0, S\rangle \\ \text{subject to} \quad & \langle A_i, S\rangle + c_i \leq 0,\ i = 1, \ldots, p, \\ & S \succeq 0, \operatorname{rank}(S) = 1. \end{aligned}$$

Since the nonconvexity is captured by $\operatorname{rank}(S) = 1$, dropping this rank constraint leads to a standard SDP. This process is also known as Shor's relaxation. □

We notice that (Primal) can be viewed as a QCQP with a quadratic coupling in the matrix variable

$$V = \begin{bmatrix} I \\ K \end{bmatrix} X^{1/2} \in \mathbb{R}^{(n+m)\times n}. \tag{19}$$

Indeed, we have $\langle Q + K^{\mathsf{T}}RK, X\rangle = \langle Q, X\rangle + \langle R, KXK^{\mathsf{T}}\rangle = \left\langle \operatorname{diag}(Q, R), VV^{\mathsf{T}}\right\rangle$ for the cost, and

$$\begin{bmatrix} A + BK \\ I \end{bmatrix} X \begin{bmatrix} A + BK \\ I \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} VV^{\mathsf{T}} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^{\mathsf{T}}.$$

This implies that the LQR problem in (Primal) may be viewed as a *Matrix Quadratic Programming* (MQP) [23], which involves a quadratic cost and constraints in matrices.

Similar to Example 1, we introduce a new variable $Z = VV^{\mathsf{T}} \in \mathbb{S}_+^{n+m}$ from the matrix variable V in (19) with $(K, X) \in \mathbb{R}^{m\times n} \times \mathbb{S}_+^n$. By definition, we have $Z \in \mathbb{S}_+^{m+n}$ and $\operatorname{rank}(Z) \leq n$. We can thus relax (Primal) into the following rank-constrained SDP:

$$\begin{aligned} \min_{Z \succeq 0} \quad & \langle \operatorname{diag}(Q, R), Z\rangle \\ \text{subject to} \quad & \Psi\left(\begin{bmatrix} A & B \\ I & 0 \end{bmatrix} Z \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^{\mathsf{T}}\right) + W = 0, \\ & \operatorname{rank}(Z) \leq n. \end{aligned} \tag{20}$$

This matrix Z may be interpreted as the following covariance matrix [20] as:

$$Z = \mathbb{E}\left[\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^{\mathsf{T}} dt\right] \quad \text{or} \quad \mathbb{E}\left[\sum_{t=0}^\infty \begin{bmatrix} x_t \\ u_t \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}^{\mathsf{T}}\right].$$

Unless $W \succ 0$, (20) generally a relaxation and not an exact reformulation of (Primal) since the matrix V in (19) has a further structure.[2] As Ψ is a linear operator, the nonconvexity of (20) solely arises from $\operatorname{rank}(Z) \leq n$. Hence, dropping the rank constraint gives a convex SDP relaxation for the primal LQR (Primal):

$$\begin{aligned} p^\star_{\text{sdp}} = \min_{Z \succeq 0} \quad & \langle \operatorname{diag}(Q,R), Z\rangle \\ \text{subject to} \quad & \Psi\left(\begin{bmatrix} A & B \\ I & 0\end{bmatrix} Z \begin{bmatrix} A & B \\ I & 0\end{bmatrix}^{\mathsf{T}}\right) + W = 0, \end{aligned} \tag{21}$$

Interestingly, the Lagrange dual of this SDP (21) is the same as (Dual). We have the following result.

Proposition 2: Consider the primal and dual LQR problems (Primal) and (Dual), and the SDP relaxation (21). Then, 1) (Dual) is a Lagrange dual problem of (21); 2) their optimal values satisfy $d^\star \leq p^\star_{\text{sdp}} \leq p^\star$.

Proof: The first statement follows from a straightforward computation of the Lagrangian of (21). As for the second statement, $d^\star \leq p^\star_{\text{sdp}}$ follows from standard weak duality in SDPs, and $p^\star_{\text{sdp}} \leq p^\star$ is valid from the rank relaxation. ■

B. Proofs of Theorem 1 and Proposition 1

It turns out that the SDP relaxation (21) and the dual LQR (Dual) also satisfy strong duality, and we can further explicitly characterize their primal-dual optimal solutions.

Lemma 2: Consider the SDP relaxation (21) and the dual LQR (Dual). Suppose Assumption 1 holds. Then, we have:

1) The duality gap between (21) and (Dual) is zero, i.e., strong duality holds with $p^\star_{\text{sdp}} = d^\star$.
2) An optimal dual solution to (Dual) is given by the unique stabilizing solution $P^\star \succeq 0$ to the ARE (15).
3) The optimal primal solution to (21) is unique, given by
$$Z^\star = \begin{bmatrix} I \\ K^\star + \Delta\end{bmatrix} X^\star \begin{bmatrix} I \\ K^\star + \Delta\end{bmatrix}^{\mathsf{T}}, \tag{22}$$
where $K^\star$ is given in (16), $X^\star$ that solves the Lyapunov equation (7) for $K^\star$, and Δ is any $m \times n$ matrix satisfying $\Delta X^\star = 0$.

When $W \succ 0$, the solution $P^\star$ to (Dual) is unique, and the factorization (22) is unique with $\Delta = 0$.

This result follows from the celebrated comparison theorem for (continuous- and discrete- time) AREs [21] and the KKT condition [7]. While [8, Th. 1] established a similar result for $W = I$, this lemma further ensures that solution $Z^\star$ to the primal SDP (21) is unique even for $W \not\succ 0$. We provide the proof details in Section IV-C.

With Lemma 2, we are now ready to prove both Theorem 1 and Proposition 1.

Proof of Theorem 1. Recall that from Proposition 2 that the optimal value of (Primal) satisfies $p^\star_{\text{sdp}} \leq p^\star$. Since an optimal solution $Z^\star$ to (21) is of the form (22), we have

$$p^\star \leq \langle Q + (K^\star)^{\mathsf{T}} R K^\star, X^\star\rangle = \langle \operatorname{diag}(Q,R), Z^\star\rangle = p^\star_{\text{sdp}} \tag{23}$$

where we use $(K^\star, X^\star)$ as a feasible solution to (Primal). Thus, (21) is tight, i.e., $p^\star = p^\star_{\text{sdp}}$. Combining this with $p^\star_{\text{sdp}} = d^\star$, we conclude the strong duality between (Primal) and (Dual), i.e., $p^\star = d^\star$. □

[2]When $W \succ 0$, one can replace $X \succeq 0$ in (Primal) by $X \succ 0$, and keep in (20) this constraint as $Z_{11} \succ 0$. In this case, we can explicitly guarantee that their feasible regions are exactly equal.

Proof of Proposition 1. The first statement of Proposition 1 is the same as the second statement of Lemma 2. The second statement of Proposition 1 essentially follows from (23), as the feasible point $(K^\star, X^\star)$ achieves the optimal value $p^\star$. This pair $(K^\star, X^\star)$ is an optimal solution to (Primal). When $W \succ 0$, the dual optimal solution is unique by Lemma 2. Moreover, we also have $X^\star \succ 0$, which implies that Δ satisfying $\Delta X^\star = 0$ is 0. Then, the factorization of $Z^\star$ in (22) is unique, leading to the uniqueness of $(K^\star, X^\star)$ by the construction of (21). □

Remark 3: For nonconvex QCQPs with a single inequality constraint, we have the famous *S-lemma*, i.e., the strong duality and tightness of Shor's SDP relaxation hold under a mild assumption. Our analysis shows that (Primal) can be viewed as an MQP [23]. One interesting direction is to address more advanced nonconvex control problems from an MQP perspective (e.g., using an extension of S-lemma [23]). □

C. Proof of Lemma 2

We finally prove the three statements in Lemma 2. This requires classical results in continuous- and discrete-time AREs, as well as careful KKT analysis between (21) and (Dual).

We first introduce two technical lemmas on the ARE (17).

Lemma 3 (Comparison theorem [21, Th. 9.1.1, Th. 13.1.1]): Suppose Assumption 1 holds. Then, the stabilizing solution $P^\star$ to the ARE (17) is unique and maximal, i.e., it satisfies $P^\star \succeq P$ for any $P \in \mathbb{S}^n$ satisfying $U(P) \succeq 0$. This stabilizing solution $P^\star$ is also positive semidefinite.

Lemma 4: Suppose Assumption 1 holds. If there exists a positive semidefinite solution $\hat{P} \succeq 0$ to the ARE (17), there exists a matrix $P_+ \in \mathbb{S}^n$ such that $U(P_+) \succ 0$,

The continuous-time case in Lemma 4 is from [24, Th. 2.23]; see Appendix C for details in the discrete-time case.

Proof of statements 1) and 2). According to [7, Th. 3.1], for establishing the first statement, it suffices to confirm $|d^\star| < \infty$ and the strict feasibility of (Dual), i.e., the optimal value of (Dual) is finite and $U(P) \succ 0$ for some $P \in \mathbb{S}^n$. We can verify the former condition $|d^\star| < \infty$ by Lemma 3, which further proves the second statement. Since $P^\star$ is maximal and $W \succeq 0$, this solution achieves the largest value among all feasible points in (Dual). Thus, we have proved the second statement in Lemma 2, which also implies $0 \leq d^\star = \langle P^\star, W\rangle < \infty$.

Next, to show dual strict feasibility, Lemma 4 guarantees the existence of some $P \in \mathbb{S}^n$ satisfying $U(P) \succ 0$. This lemma indeed ensures the strict feasibility of (Dual). Consequently, [7, Th. 3.1] ensures the first statement, i.e., the strong duality holds. The second statement is clear from the above argument.

Proof of statement 3) By the strong duality in the first statement, [7, Th. 3.1] further confirms that a minimizer $Z^\star \in \mathbb{S}^{n+m}_+$ to (21) exists. Now, any such $Z^\star$ is characterized by the following KKT condition between (21) and (Dual):

$$\Psi\left(\begin{bmatrix} A & B \\ I & 0\end{bmatrix} Z^\star \begin{bmatrix} A & B \\ I & 0\end{bmatrix}^{\mathsf{T}}\right) + W = 0,\ Z^\star \succeq 0, \tag{24a}$$

$$U(P^\star) \succeq 0,\ P^\star \in \mathbb{S}^n, \quad (24b) \qquad \langle Z^\star, U(P^\star)\rangle = 0, \quad (24c)$$

where $P^\star$ solves (Dual). From the statement 2) of Lemma 2, the stabilizing solution $P^\star \succeq 0$ to the ARE (15) constitutes a solution to (Dual). Now, recall (15) for $P^\star$, i.e., $U_{11}(P^\star) -$

$U_{12}(P^\star)U_{22}^{-1}(P^\star)U_{12}^\top(P^\star) = 0$. Substituting this into the complementary slackness (24c) yields

$$0 = \langle Z^\star, U(P^\star)\rangle = \left\langle Z^\star, \begin{bmatrix} U_{11}(P^\star) & U_{12}(P^\star) \\ U_{12}^\top(P^\star) & U_{22}(P^\star)\end{bmatrix}\right\rangle \tag{25}$$

$$= \mathbf{Tr}\left(Z^\star \begin{bmatrix} U_{12}(P^\star)U_{22}^{-1/2}(P^\star) \\ U_{22}^{1/2}(P^\star)\end{bmatrix}\begin{bmatrix} U_{12}(P^\star)U_{22}^{-1/2}(P^\star) \\ U_{22}^{1/2}(P^\star)\end{bmatrix}^\top\right)$$

from the ARE (15) with $P = P^\star$ and the factorization

$$U(P^\star) = \begin{bmatrix} U_{12}(P^\star)U_{22}^{-1}(P^\star)U_{12}^\top(P^\star) & U_{12}(P^\star) \\ U_{12}^\top(P^\star) & U_{22}(P^\star)\end{bmatrix}$$

$$= \begin{bmatrix} U_{12}(P^\star)U_{22}^{-1/2}(P^\star) \\ U_{22}^{1/2}(P^\star)\end{bmatrix}\begin{bmatrix} U_{12}(P^\star)U_{22}^{-1/2}(P^\star) \\ U_{22}^{1/2}(P^\star)\end{bmatrix}^\top.$$

Note that (24c) implies $\mathrm{rank}(Z^\star)+\mathrm{rank}(U(P^\star)) \le n+m$. Thus, combining this with $\mathrm{rank}(U(P^\star)) = \mathrm{rank}(U_{22}(P^\star)) = \mathrm{rank}(R) = m$ (from (13) and $P^\star \succeq 0$), we obtain $\mathrm{rank}(Z^\star) \le n$, which enforces $Z^\star$ to be of the form

$$Z^\star = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^\top \tag{26}$$

for $V_1 \in \mathbb{R}^{n\times n}$, $V_2 \in \mathbb{R}^{m\times n}$. Substituting this into (25) gives

$$\langle Z^\star, U(P^\star)\rangle = \left\| \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^\top \begin{bmatrix} U_{12}(P^\star)U_{22}^{-1/2}(P^\star) \\ U_{22}^{1/2}(P^\star)\end{bmatrix}\right\|_F^2$$

$$= \| V_1^\top U_{12}(P^\star)U_{22}^{-1/2}(P^\star) + V_2^\top U_{22}^{1/2}(P^\star)\|_F^2 = 0.$$

Hence, we obtain $V_1^\top U_{12}(P^\star)U_{22}^{-1/2}(P^\star) + V_2^\top U_{22}^{1/2}(P^\star) = 0$, which implies $V_2 = -\left(U_{22}(P^\star)\right)^{-1}U_{12}^\top(P^\star)V_1 = K^\star V_1$ from (16). We know from (26) that $Z^\star$ must be of the form

$$Z^\star = \begin{bmatrix} V_1 \\ K^\star V_1 \end{bmatrix}\begin{bmatrix} V_1 \\ K^\star V_1 \end{bmatrix}^\top = \begin{bmatrix} I \\ K^\star \end{bmatrix} V_1V_1^\top \begin{bmatrix} I \\ K^\star \end{bmatrix}^\top. \tag{27}$$

The construction of (21) indicates that (24a) is reduced to the Lyapunov equation (7) for $K = K^\star$, and $V_1V_1^\top$ is uniquely determined as $V_1V_1^\top = X^\star$. Hence,the primal problem is uniquely solved by $Z^\star$ in (22).

Proof of the uniqueness when $W \succ 0$: First, we verify that $P^\star$ from the ARE (15) uniquely solves (Dual). Indeed, if we have two different solutions as $\langle \hat{P}, W\rangle = \langle P^\star, W\rangle$ for $\hat{P} \neq P^\star$, we have $\mathbf{Tr}((P^\star - \hat{P})W) = 0$, and thus

$$(P^\star - \hat{P})W = 0$$

from $P^\star \succeq \hat{P}$ (Recall that $\mathbf{Tr}(AB) = 0$ for $A, B \succeq 0$ implies $AB = 0$). However, this is a contradiction because $P^\star - \hat{P} \neq 0$. Hence, the dual problem has the unique solution $P^\star$.

Finally, $W \succ 0$ implies $X^\star \succ 0$, so that $\Delta X^\star = 0$ yields $\Delta = 0$. Hence, the factorization in (26) is also unique.

Remark 4: In our proof, we identify the form of $K^\star$ in a constructive manner by using the rank structure as in (25), (26), and (27). Namely, we do not need the knowledge of the optimal gain for the strong duality. This aspect may be viewed as a benefit, compared with the earlier results [6], [15]. □

V. Conclusion

We presented a unified duality analysis for continuous-time and discrete-time LQRs. Using the linear operator Ψ, we formulated a common nonconvex primal problem for both time domains. Deriving the Lagrange dual, we demonstrated the strong duality and optimal solutions that recovered the LQR optimal gains and AREs. Moreover, we showed that the common formulation admits a tight SDP relaxation with the same dual problem, highlighting an alternative aspect of LQR as a well-behaved QCQP.

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Appendix

A. Derivation of the adjoint operator Ψ^*

We can derive the adjoint operators from a simple algebra. Specifically, for a block matrix

$$Z = \begin{bmatrix} F & G \\ G^\top & H \end{bmatrix} \in \mathbb{S}^{2n},$$

in the continuous-time case, we have

$$\Psi_{\mathtt{ct}}(Z) = G + G^\top.$$

Then, for any $P \in \mathbb{S}^n$, it is not difficult to see that

$$\langle \Psi_{\mathtt{ct}}(Z), P\rangle = \langle G + G^\top, P\rangle = \left\langle Z, \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \right\rangle.$$

Hence,

$$\Psi^*_{\mathtt{ct}}(P) = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}.$$

Similarly, for the discrete-time case,

$$\Psi_{\mathrm{dt}}(Z) = F - H,$$

we have

$$\langle \Psi_{\mathtt{dt}}(Z), P\rangle = \langle F - H, P\rangle = \left\langle Z, \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \right\rangle,$$

so that

$$\Psi^*_{\mathrm{dt}}(P) = \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix}.$$

B. Proof of Lemma 1

Here, we show $d(P) = -\infty$ if $U(P) \not\succeq 0$. First, if $U(P)$ is not positive semidefinite, there exists an eigenvector $v = [v_1^\top, v_2^\top]^\top \neq 0$ such that $U(P)v = -\lambda v$ with some $\lambda > 0$.

We separately show the two cases, i.e., $v_1 \neq 0$ and $v_1 = 0$. When $v_1 \neq 0$, we can choose

$$X = \eta v_1 v_1^\top \succeq 0, \quad K = v_2 v_1^\top$$

with $\eta > 0$. In addition, we can set $\|v_1\| = 1$ without loss of generality. Direct computations now ensure that

$$\begin{bmatrix} I \\ K \end{bmatrix} X \begin{bmatrix} I \\ K \end{bmatrix}^\top = \eta \begin{bmatrix} v_1 v_1^\top & v_1 v_2^\top \\ v_2^\top v_1 & v_2 v_2^\top \end{bmatrix} = \eta v v^\top,$$

which gives $L(K, X, P) = \langle P, W\rangle - \eta\lambda\|v\|^2$ from (11). We now see $d(P) = -\infty$ as $\eta \to \infty$.

On the other hand, if $v_1 = 0$, we have $v_2 \neq 0$ by $v \neq 0$. In addition, we have

$$v_2^\top U_{22}(P) v_2 = -\lambda\|v_2\|^2 < 0$$

for the bottom-right block of $U(P)$. We now choose

$$X = w w^\top \succeq 0, \quad K = \eta v_2 w^\top$$

with a unit vector w and $\eta > 0$. With this choice, it is not difficult to verify that

$$L(K, X, P) = \langle P, W\rangle + w^\top U_{11}(P) w - \eta^2 \lambda \|v_2\|^2,$$

where $U_{11}(P)$ is the top-left block of $U(P)$. Consequently, we obtain $d(P) = -\infty$ by $\eta \to \infty$. This completes the proof.

C. Proof of Lemma 4 for discrete-time systems

Here, we prove that $U(P_+) \succ 0$ for some $P_+ \in \mathbb{S}^n$. The continuous-time case follows from [24, Th. 2.23]. One can also verify the discrete-time case in a similar manner. For completeness, we provide the proof below.

Let $Q(\epsilon) = Q - \epsilon I$. Note that $(Q(\epsilon), A)$ is still detectable for any $\epsilon \in [0, \bar{\epsilon})$ with a sufficiently small $\bar{\epsilon} > 0$. Thus, the ARE with $Q(\epsilon)$

$$A^\top P A - P + Q(\epsilon) - A^\top P B (R + B^\top P B)^{-1} B^\top P A = 0 \quad (28)$$

has a symmetric solution $P = P_\epsilon$ [21]. Further, [25, Th. 2.1] ensures that P_ϵ continuously depends on ϵ in a sufficiently small neighborhood of $\epsilon = 0$. Then, since

$$R + B^\top P_0 B \succ 0$$

holds from $P^\star = P_0 \succeq 0$ and $R \succ 0$, there exist a sufficiently small $\hat{\epsilon} \in (0, \bar{\epsilon})$ and a symmetric matrix $P_+ =, P_{\hat{\epsilon}} \in \mathbb{S}^n$ such that we have both

$$R + B^\top P_{\hat{\epsilon}} B \succ 0$$

and (28), i.e.,

$$A^\top P_{\hat{\epsilon}} A - P + Q(\hat{\epsilon}) - A^\top P_{\hat{\epsilon}} B (R + B^\top P_{\hat{\epsilon}} B)^{-1} B^\top P_{\hat{\epsilon}} A = 0.$$

Hence, by

$$A^\top P_{\hat{\epsilon}} A - P + Q - A^\top P_{\hat{\epsilon}} B (R + B^\top P_{\hat{\epsilon}} B)^{-1} B^\top P_{\hat{\epsilon}} A = \hat{\epsilon} I \succ 0,$$

the Schur complement yields

$$U(P_{\hat{\epsilon}}) = \begin{bmatrix} A^\top P_{\hat{\epsilon}} A - P + Q & A^\top P_{\hat{\epsilon}} B \\ B^\top P_{\hat{\epsilon}} A & R + B^\top P_{\hat{\epsilon}} B \end{bmatrix} \succ 0,$$

which completes the proof.